DIOPHANTINE ANALYSIS.

82. **Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.**

In the series $1^3+3^3+5^3$.... find n so that the nth term and the sum of n terms shall both be squares.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The conditions of the problem require $1^3+3^3+5^3+\ldots\ldots+(2n-1)^3=n^2(2n^2-1)=\square$, and $(2n-1)^3=\square$.

Whence it follows that $2n-1=\square$, and $2n^2-1=\square$.

$2n-1=\square$, when $n=r^2+(r-1)^2$, as 1, 5, 13, 25, 41, etc.

Hence $2n^2-1=2[r^2+(r-1)^2]^2-1=\square$.

Whence $2[r^2+(r-1)^2]^2=\square+1$, two times a square equals the sum of two squares, the general formula for which is $2(p^2+q^2)^2=[(p+q)^2-2q^2]^2+[(p-q)^2-2q^2]^2$.

Then $p^2+q^2=r^2+(r-1)^2\ldots.(1)$, and $(p-q)^2-2q^2=\pm1\ldots.(2)$.

From (1), put $p=r$ and $q=(r-1)$; and substituting in (2) we obtain $1-2(r-1)^2=\pm1$, or $2(r-1)^2=0$ or 2.

Whence $r-1=0$ or ±1, and $r=1$ or 2 or 0.

Substituting these values of r in $n=r^2+(r-1)^2$, we find $n=1$ and 5, apparently the only integral values.

When $n=1$, $2n-1=1$, $n^2(2n^2-1)=1$, and $(2n-1)^3=1$.

When $n=5$, $2n-1=9=3^2$, $n^2(2n^2-1)=1^3+3^3+5^3+7^3+9^3=35^2$, and $(2n-1)^3=(3^2)^3=(3^3)^2=27^2$.

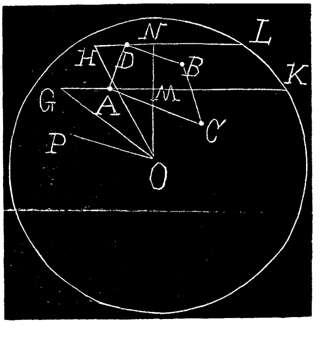
Also solved by *G. B. M. ZERR, J. SCHEFFER, COOPER D. SCHMITT*, and the *PROPOSER*.

AVERAGE AND PROBABILITY.

88. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.**

Find the average volume of the tetrahedron formed by joining four random points in a sphere.

Solution by the PROPOSER.



Let GK be the diameter of the section of the sphere made by a plane through the three random points A, B, C; HL the diameter of a parallel section through the fourth random point D; M, N the centers of these sections, respectively; O the center of the sphere; OP a line such that AB is always parallel to the plane MOP.

Let $OG=OH=r$, $MA=x$, $AB=y$, $AC=z$, $DN=u$, $\angle GOM=\theta$, $\angle BAM=\varphi$, $\angle CAM=\psi$, $\angle HON=\beta$, $\angle MOP=\lambda$, and the angle the plane POM makes with a fixed plane through $OP=\rho$.

An element of the sphere at A is $r\sin\theta d\theta 2\pi x dx$; at B, $y^2 dy d\varphi d\lambda$; at C, $\sin(\varphi+\psi)\sin\lambda z^2 dz d\psi d\rho$; at D, $r\sin\beta d\beta 2\pi u du$.

The limits of θ are 0 and π; of β, 0 and θ; of x, 0 and $r\sin\theta=x'$ and tripled; of u, 0 and $r\sin\beta=u'$ and sextupled; of φ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\varphi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and π; of y, 0 and $2x\cos\varphi=y'$; of z, 0 and $2x\cos\psi=z'$.

The area of the triangle $ABC=\frac{1}{2}yz\sin(\varphi+\psi)$.

Altitude of tetrahedron$=r(\cos\beta-\cos\theta)$.

Volume of tetrahedron$=\frac{1}{6}ryz\sin(\varphi+\psi)(\cos\beta-\cos\theta)$.

Since the whole number of ways four points can be taken is $(\frac{4}{3}\pi r^3)^4$, the required average triangle is

$$\Delta=\frac{729}{128\pi^4 r^{12}}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^\pi\int_0^\pi\int_0^{y'}\int_0^{z'}\tfrac{1}{6}ryz\sin(\varphi+\psi)$$

$$\times(\cos\beta-\cos\theta)r\sin\theta d\theta 2\pi x dx r\sin\beta d\beta 2\pi u du\sin(\varphi+\psi)d\varphi d\psi\sin\lambda d\lambda d\rho y^2 dy z^2 dz$$

$$=\frac{243}{16\pi^2 r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^\pi\int_0^\pi\int_0^{y'}\sin\theta x^5\sin\beta u\sin^2(\varphi+\psi)\cos^4\psi$$

$$\times\sin\lambda(\cos\beta-\cos\theta)d\theta d\beta dx du d\varphi d\psi d\lambda d\rho y^3 dy$$

$$=\frac{243}{4\pi^2 r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^\pi\int_0^\pi\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u\sin^2$$

$$\times(\varphi+\psi)\cos^4\varphi\cos^4\psi\sin\lambda d\theta d\beta dx du d\varphi d\psi d\lambda d\rho$$

$$=\frac{243}{2\pi r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u\sin^2(\varphi+\psi)$$

$$\times\cos^4\varphi\cos^4\psi d\theta d\beta dx du d\varphi d\psi$$

$$=\frac{81}{64\pi r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u(15\pi\cos^4\varphi$$

$$-12\pi\cos^6\varphi+30\varphi\cos^4\varphi-24\varphi\cos^6\varphi-4\sin\varphi\cos^7\varphi+30\sin\varphi\cos^5\varphi)d\theta d\beta dx du d\varphi$$

$$=\frac{1215\pi}{256r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u d\theta d\beta dx du$$

$$=\frac{1215\pi}{512r^7}\int_0^\pi\int_0^\theta\int_0^{x'}\sin\theta\sin^3\beta(\cos\beta-\cos\theta)x^9 d\theta d\beta dx$$

$$=\frac{243\pi r^3}{1024}\int_0^\pi\int_0^\theta\sin^{11}\theta\sin^3\beta(\cos\beta-\cos\theta)d\theta d\beta$$

$$=\frac{81\pi r^3}{4096}\int_0^\pi\sin^{11}\theta(8-4\sin^2\theta-\sin^4\theta-8\cos\theta)d\theta=\frac{36\pi r^3}{715}.$$